SDEs and filtering

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Overview

Stochastic Differential Equations - SDEs

- Ideas
- Technical properties
- Itô formula

Filtering - Partially observed Models

- Optimal filtering
- Kalman filters
- SMC filters

General

Generalize classic ODE

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \mu(t, X_t)$$

to stochastic ODE

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \mu(t, X_t) + \sigma(t, X_t)v_t,$$

where v_t is a formal time derivative of a continuous time random walk.

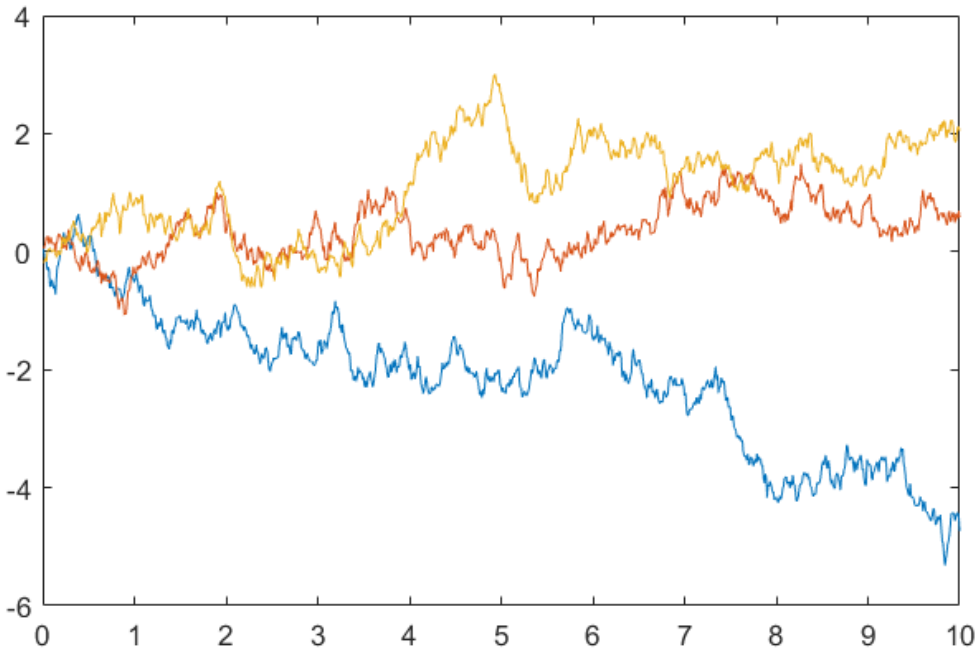
Such object cannot be defined!!!

This is an applied course. We try to preserve the relevant properties.

Derivative of a standard Brownian motion

A **standard Brownian motion** is a stochastic process satisfying

1. $W(0) = 0$ *a.s.*
2. $W(t)$ is continuous w.p.1.
3. $W(t) - W(s)$ is indep. of $W(u) - W(0)$ for $t > s \geq u > 0$
4. $W(t) - W(s) \sim N(0, t - s)$



Other types of continuous time random walks

- Poission process $N(t)$ or compensated Poisson process $N(t) - \lambda t$. It can be defined as $Y(t) = \sum_{n=1}^{N(t)} 1$. (Note the convention $\sum_{n=1}^{0} 1 \triangleq 0$)
- Compound Poisson process $Y(t) = \sum_{n=1}^{N(t)} \xi_n$, where ξ_n are *iid* r.v.s
- Levy processes, $L(\mu, \sigma, \nu)$, generalizing Brownian motion and infinite activity compound Poisson processes, where μ is the drift, σ the diffusion and ν is a jump measure.

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Derivative of a Brownian motion

- Try to use the formal definition of a derivative

$$v(t) = \frac{\mathrm{d}W}{\mathrm{d}t}(t) = \lim_{h \to 0} \frac{W(t+h) - W(t)}{h} \tag{1}$$

- Note that $\frac{W(t+h)-W(t)}{h} \in N\left(0, \frac{h}{h^2}\right)$.
- **Thus, the derivative process would have zero mean and infinite variance!**
- We have to define Stochastic Differential Equations carefully to obtain a meaningful mathematical object.

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Interpretation of μ and σ

We are interested in properties when modelling using SDEs

- $\mu = \lim_{h \to 0} \frac{1}{h} \mathbf{E}[X_{t+h} - X_t | X_t]$.
- $\sigma\sigma^T = \lim_{h \to 0} \frac{1}{h} Var[X_{t+h} - X_t | X_t]$.
- Thus $\frac{\mathrm{d}X}{\mathrm{d}t} = \mu(t, X_t) + "noise"$,
- where $\mathbf{E}[noise] = 0$ *regardless of* $\sigma(\cdot)$. Almost true...

These choices conditions have far-reaching consequences!

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Problems (P) and solutions (S)

- P: The time derivative of a BM cannot be defined, $\nexists v_t$
- S: Thus, we define stoch. integral equations instead

$$\int \mathrm{d}X_u = \int \mu(u, X_u)\mathrm{d}u + \int \sigma(u, X_u)\mathrm{d}W_u \qquad (2)$$

- P: The Brownian Motion has unbounded variation - the Riemann integral does not converge.
- S: But the BM has bounded quadratic variation. Thus, define the stoch. integral as limit in $L^2(\mathrm{d}\mathbb{P} \times \mathrm{d}t)$.

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Length of Brownian motion

Heuristic argument - consider a partition of time $t = u_0 < u_1 < \ldots < u_N = T$, with $\delta = T/N$.

- We now have N elements, each of length $\sqrt{\delta^2 + (\delta W)^2} = \ldots = \mathcal{O}_p(\sqrt{\delta})$ as $\delta \to 0$
- Total length is $N\mathcal{O}_p(\sqrt{\delta}) = T\mathcal{O}_p(\sqrt{N})$

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Integrals

Define a partition in time $t = u_0 < u_1 < \ldots < u_N = T$ and let $N \to \infty$

- $\int \mu(u, X_u)\mathrm{d}u$ is a standard **Riemann integral**.
- $\int \mu(u, X_u)\mathrm{d}u = \lim_{N\to\infty} \sum \mu(\tau, X_\tau)(u_{n+1} - u_n)$, where $u_n \leq \tau \leq u_{n+1}$.

- $\int \sigma(u, X_u)\mathrm{d}W_u$ is a **Itô integral**.
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Properties of stochastic integrals

The **filtration** $\mathcal{F}_t$ is the sigma algebra "information" generated by the Brownian motion $W_u,\ u \leq t$.

- Martingale property of stochastic integrals
  $\mathbf{E}\left[\int_0^T \sigma(W_u)\mathrm{d}W_u|\mathcal{F}_t\right] = \int_0^t \sigma(W_u)\mathrm{d}W_u + 0.$
- Ito isometry $\mathbf{E}\left[\left(\int_0^T \sigma(W_u)\mathrm{d}W_u\right)^2\right] = \int_0^T \mathbf{E}\left[\sigma^2(W_u)\right]\mathrm{d}u$

Proof of Martingale property

Exercise

- **To do:** Calculate $\int W_u \mathrm{d}W_u$.
- **To do:** Calculate $\mathbf{E}[\int_t^T W_u \mathrm{d}W_u | \mathcal{F}_t]$.

Finding analytical solutions to SDEs

3 well known cases

- ABM $\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$ Solution: $X_t = X_0 + \mu t + \sigma W_t$
- GBM $\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$ Solution: $X_t = X_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right)$
- Ornstein-Uhlenbeck $dX_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma dW_t$ Gaussian process, AR(1)

Exercise: Solve the ABM using the definitions.

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Chain rule for SDEs

- Analytical solutions to ODEs are solved by transforming, and thereafter integrating the system.
- The same holds for SDEs.
- Let $\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t$
- Introduce $Y_t = F(t, X_t)$. What about $\mathrm{d}Y_t$?
- It can be shown that for $F \in C^{1,2}$

$$\mathrm{d}Y_t = \left(F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right)\mathrm{d}t + \sigma F_x \mathrm{d}W_t \tag{3}$$

$$= (F_t + \mathcal{A}F)\,\mathrm{d}t + \sigma F_x \mathrm{d}W_t. \tag{4}$$

Look at handouts for more theoretical background.

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Monte Carlo methods

We can easily simulate from an SDE.

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t \tag{5}$$

Convergence is defined as

- **Weak convergence** of $\mathcal{O}(h^\beta)$ is defined for $0 < h < h_0$ for some small h_0

$$\left|\mathbf{E}[g(X_T)] - \mathbf{E}[g(X_T^h)]\right| \leq Dh^\beta \tag{6}$$

- **Strong convergence** of $\mathcal{O}(h^\alpha)$ is defined as

$$\mathbf{E}\left[\sup\left|X_T - X_T^h\right|\right] \leq Ch^\alpha \tag{7}$$

Popular schemes

- The *Euler-Maruyama* scheme is given by

$$X_{t+h} = X_t + \mu(t, X_t)h + \sigma(t, X_t)\delta W_t^h \tag{8}$$

  where $\delta W_t^h = W_{t+h} - W_t$.
- Weak convergence $\mathcal{O}(h)$, strong convergence $\mathcal{O}(\sqrt{h})$
- The *Milstein* scheme is given by

$$\begin{aligned} X_{t+h} = X_t + \mu(t, X_t)h + \sigma(t, X_t)\delta W_t^h + \\ \frac{1}{2}\sigma(t, X_t)\sigma'(t, X_t)\left((\delta W_t^h)^2 - h\right) \end{aligned} \tag{9}$$

- Weak convergence $\mathcal{O}(h)$, strong convergence $\mathcal{O}(h)$. Note - difficult to generalize to higher dimensions!

Fokker-Planck - Likelihood estimation

The transition density $p(x_t|x_s)$ is derived from the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(x_t|x_s) = \mathcal{A}^* p(x_t|x_s) \tag{10}$$

where $\mathcal{A}^*$ is the Hermitian adjoint operator for the Ito generator $\mathcal{A}$

$$\mathcal{A}^* p = -\left(\frac{\partial}{\partial x}\mu(\cdot)p\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(\cdot)p\right) \tag{11}$$

Thus, the transition density is found by solving this PDE with initial condition $p(x_s) = \delta(x - x_s)$.

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Monte Carlo - Likelihood estimation

A simple method is to use kernel density estimation A more refined version is the *Pedersen estimator*

$$p(x_t|x_0) = \int p(x_t, x_s|x_0)\mathrm{d}x_s, \ 0 < s < t \tag{12}$$

$$= \int p(x_t|x_s)p(x_s|x_0)\mathrm{d}x_s \tag{13}$$

$$= \mathbf{E}\left[p(x_t|x_s)|\mathcal{F}_0\right] \tag{14}$$

This is easy to compute using Monte Carlo methods and is not biased, contrary to kernel density estimation.

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Evolution of mobile phones



The phone on the left is the Motorola DanTac 8000X, the first "mobile" phone (weight about 2 kg) and costing about 4 000 USD when launched in 1984.

Dynamical model of phone market

Carry capacity

$$\mathrm{d}C = \mu C \mathrm{d}t + \sigma C \mathrm{d}W_t^C \tag{15}$$

Simulation using four competing brands (subject to carry capacity)
Each brand is modeled using standard growth model, with parameters capturing the growth and adaptivity properties of specific companies.

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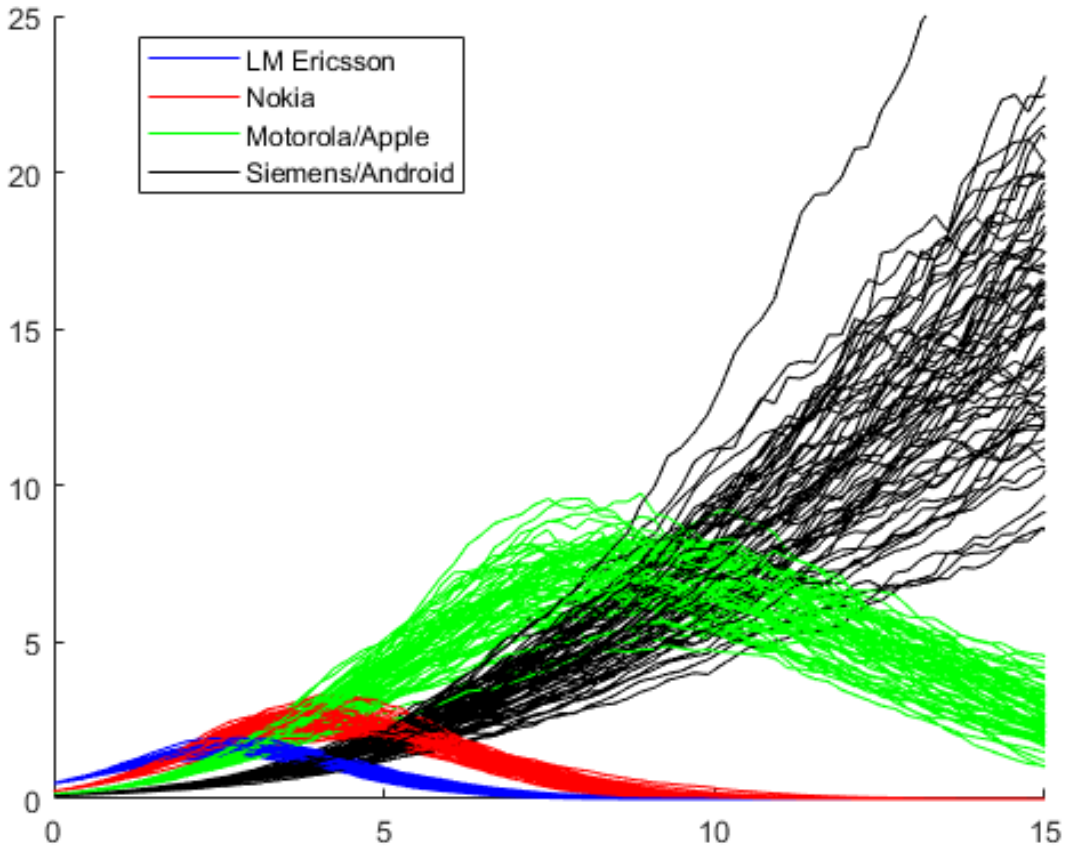
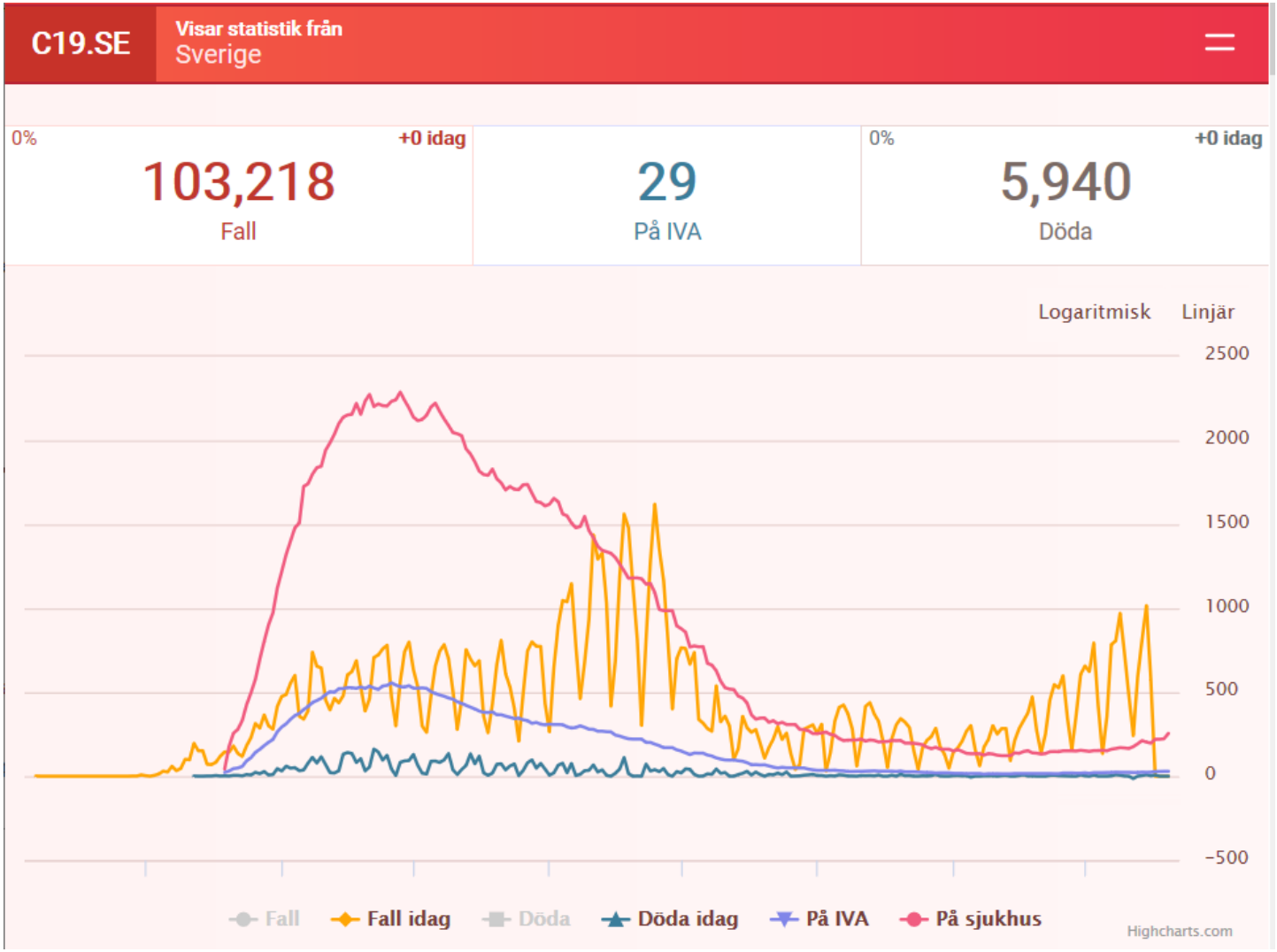
Evolution

Growth over time, 20 different realizations



`Matlab: GenLotkaVolterra.m`

Covid-19 cases in Sweden as of October 19th, 2020

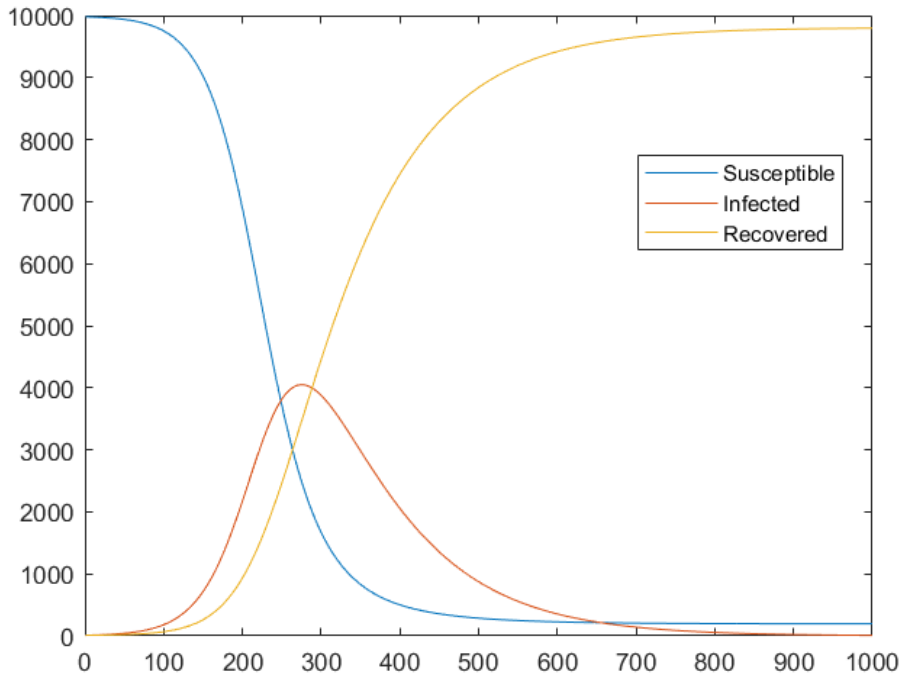


The standard model is the S-I-R models

Three components, **S**usceptibles, **I**nfected and **R**ecoved.

$$\begin{pmatrix} \mathrm{d}S_t \\ \mathrm{d}I_t \\ \mathrm{d}R_t \end{pmatrix} = \begin{pmatrix} -\beta \frac{S_t I_t}{N_t} \\ \beta \frac{S_t I_t}{N_t} - \gamma I_t \\ \gamma I_t \end{pmatrix} \mathrm{d}t \tag{16}$$

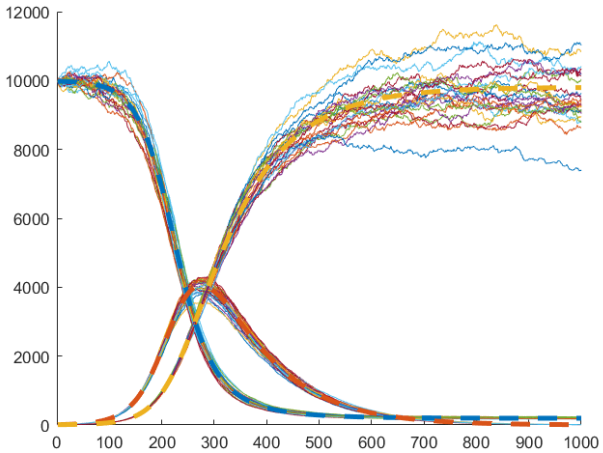
with $\beta = 0.4$ and $\gamma = 0.1$.

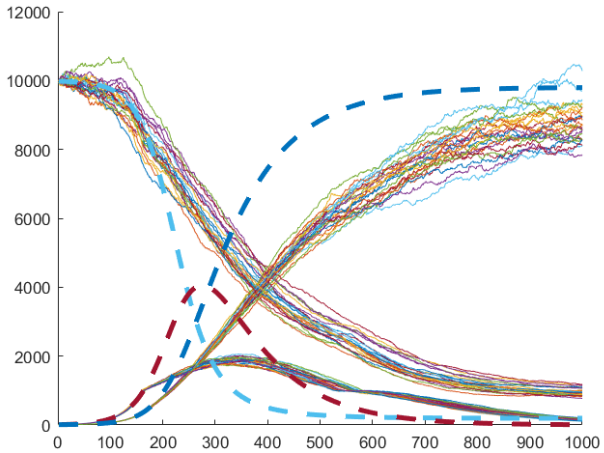


Stochastic version of that model

$$\begin{pmatrix} \mathrm{d}S_t \\ \mathrm{d}I_t \\ \mathrm{d}R_t \end{pmatrix} = \begin{pmatrix} -\beta \frac{S_t I_t}{N_t} \\ \beta \frac{S_t I_t}{N_t} - \gamma I_t \\ \gamma I_t \end{pmatrix} \mathrm{d}t + \sigma \begin{pmatrix} S_t & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & R_t \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^S \\ \mathrm{d}W_t^I \\ dW_t^R \end{pmatrix} \quad (17)$$

with $\sigma = 0.01$.





Feel free to play with the `Matlab` scripts, including flattening the curve interventions ($\beta = 0.4$ left, $\beta = 0.2$ when $I_t > 1000$ right).

How would you model Covid-19?

Same plot as before, except in log-scale - the number of cases is increasing steadily



Filtering

- Assume that we have a latent (hidden) process X, in discrete or continuous time.
- X is assumed to be Markov, with known initial distribution.
- Observations Y are generated from X
- and perturbated by noise.

$$
\begin{aligned}
x_0 &\sim p(x_0) \\
x_{n+1} &= f(t_n, x_n, e_{n+1}), && \Leftrightarrow p(x_{n+1}|x_n) \text{ known} \\
y_{n+1} &= h(t_{n+1}, x_{n+1}, \eta_{n+1}), && \Leftrightarrow p(y_n|x_n) \text{ known}
\end{aligned}
$$

Dependence structure

Consider the model

$$x_{n+1} = ax_n + e_{n+1} \tag{18}$$

$$y_{n+1} = x_{n+1} + \eta_{n+1} \tag{19}$$

Exercise: Show that the correlation structure for $\{y\}$ is that of an ARMA(1,1) model.

Likelihood function

The log-likelihood is given by

$$\ell(\theta) = \log p_\theta(y_1, \ldots, y_N) \tag{20}$$

$$= \sum_{n=1}^{N} \log p_\theta(y_n | y_1 \ldots, y_{n-1}) = \sum_{n=1}^{N} \log p_\theta(y_n | y_{1:n-1}) \tag{21}$$

This can be expanded as

$$p_\theta(y_n | y_1 \ldots, y_{n-1}) = \int p_\theta(y_n, x_n | y_{1:n-1}) \mathrm{d}x_n \tag{22}$$

$$\int \underbrace{p_\theta(y_n | x_n)}_{\textit{Known}} p_\theta(x_n | y_{1:n-1}) \mathrm{d}x_n \tag{23}$$

Filter equations

Recursive formulas can be derived for *the predictive density* $p_\theta\left(x_n|y_{1:n-1}\right)$ is given by

$$p_\theta\left(x_n|y_{1:n-1}\right) = \int \underbrace{p_\theta\left(x_n|x_{n-1}\right)}_{\textit{Known}} p_\theta\left(x_{n-1}|y_{1:n-1}\right) \mathrm{d}x_{n-1}. \tag{24}$$

and for *the filter density*

$$p_\theta(x_n|y_{1:n}) = \frac{p_\theta(y_n|x_n)p_\theta(x_n|y_{1:n-1})}{\int p_\theta(y_n|x_n)p_\theta(x_n|y_{1:n-1})\mathrm{d}x_n}, \tag{25}$$

Filter algorithms

Exact algorithms

- **HMM filter** "Trivial", all integrals are sums
- **Kalman filter** Linear, Gaussian transformations.

Approximate filters

- (Iterated) Extended Kalman filter
- Unscented Kalman filter
- Gaussian sum filters
- Ensemble Kalman filter
- Seq. Monte Carlo filters

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Kalman filter

Main reference for derivations of this type: The Matrix Cookbook[1]

$$x_{n+1} = Ax_n + e_n, \;\; e_n \sim N(0, Q) \tag{26}$$
$$y_n = Cx_n + \epsilon_n, \;\; \epsilon \sim N(0, R) \tag{27}$$

Initial conditions

- $p(x_0) = \phi(x_0; m_0, P_0)$

Predictive density, $p(x_n|y_{1:n-1}) = \phi(x_n; m_{n|n-1}, P_{n|n-1})$

- $m_{n|n-1} = Am_{n-1|n-1}$
- $P_{n|n-1} = AP_{n-1|n-1}A^T + Q$

[1] https://www2.imm.dtu.dk/pubdb/pubs/3274-full.html

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Kalman filter

Likelihood term $p(y_n|y_{1:n-1}) = \phi(y_n; \hat{y}_{n|n-1}, \Omega_n)$

- $\hat{y}_{n|n-1} = Cm_{n|n-1}$
- $\Omega_n = CP_{n|n-1}C^T + R$

Filter density

- $K_n = P_{n|n-1}C^T\Omega_n^{-1}$
- $m_{n|n} = m_{n|n-1} + K_n\left(y_n - \hat{y}_{n|n-1}\right)$
- $P_{n|n} = (I - KC)\,P_{n|n-1}$

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Approx. Gaussian Filters

Treat all distributions as Gaussians, $p(x|\cdot) = \phi(x, \cdot, \cdot)$

- Approximate Kalman filters (think QML)
- Easy to implement
- Nearly optimal (projections in Hilbert space)
- Historical reasons
- EKF is the simplest option (?), Iterated EKF is often more accurate with UKF often being even better. *Thus, you should know the EKF, be familiar with the IEKF and use the UKF*

`Matlab code: UKFERK1d.m`

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Sequential Monte Carlo Filters

Treat all distributions as countable sums of Dirac measures $p_J(x|\cdot) = \sum_j \frac{\omega_j}{\sum \omega_l} \delta(x - x_j)$

- Use when the drift is highly non-linear
- Non-Gaussian innovations
- Restricted state spaces,
- Performance is critical, and computational resources are abundant.
- E.g. Stochastic volatility

Proofs in handouts.

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- Computing moments

$$\mathbf{E}\left[\phi(x_n)|y_{1:n}\right] = \int \phi(x_n)p(x_n|y_{1:n})\mathrm{d}x_n \tag{28}$$

$$\approx \int \phi(x_n)p_J(x_n|y_{1:n})\mathrm{d}x_n \tag{29}$$

$$= \sum_j \frac{\omega_j}{\sum \omega_l}\phi(x_{n,j}) \tag{30}$$

- Moreover, prediction is easy as

$$p_J(x_{n+1}|y_{1:n}) = \sum_j \frac{\omega_j}{\sum \omega_l}\delta(x - x_{n+1,j}) \tag{31}$$

with simulated as

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Bootstrap filter

Initial conditions

- Sample $x_{0,j} \sim p(x_0)$

Prediction

- Sample $x_{n,j} \sim p(x_n|x_{n-1,j})$

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Bootstrap filter

Likelihood computation

- Assume $p_J(x_{n+1}|y_{1:n}) = \sum_j \frac{\omega_j}{\sum \omega_l} \delta(x - x_{n+1,j})$
- Then

$$p(y_{n+1}|y_{1:n}) = \int p(y_{n+1}|x_{n+1}) p_J(x_{n+1}|y_{1:n}) \mathrm{d}x_n \quad (33)$$

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Empirical Filter density

-

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- **Important:** Resample the weights.

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Parameter estimation

Still an open question

- Direct maximization is difficult for a variety of reasons
- EM
- Iterated filtering
- PMMH

Boat navigation

One possible project on this course was navigation in a lake, based on only a sonar and a chart. This film is by a former student on this course, Jonas Hallgren

`https://www.youtube.com/watch?v=bNDcXDilSHA`

Information: Red star = origin, Red triangle = true pos. of the boat, Blue star = particle, Green star-ring = current guess of location.

Feedback

Send feedback, questions and information about typos to `erik.lindstrom@matstat.lu.se`.